The method for uniform distribution of points on surfaces in multi-dimensional Euclidean space

N.P. Kopytov, E.A. Mityushov

Ural Federal University named after the first President of Russia B.N. Yeltsin, Ekaterinburg, Russia

e-mail: nikitako@mail.ru

Abstract

The problem of uniform distribution of points on surfaces in multi-dimensional Euclidean space is considered. Method for uniform distribution of points on analytic surfaces defined by the parametric method in multi-dimensional Euclidean space is proposed. The proposed method can be used for uniform distribution of points on the hypersphere and hyperellipsoid, as an additional method to already existing, and for the uniform distribution of points on the other parametric surfaces in a multidimensional Euclidean space. Due to generality the proposed method can be also used for uniform distribution of points on curves in a multi-dimensional Euclidean space. The method also works in three-dimensional physical space which is an ordinary for the human perception, and certainly it can be applied to a uniform distribution of points on curves and surfaces in three dimensional space for various scientific problems.

Key words: hypersurface points picking, choosing random points on surfaces and curves, uniform distributions of points on surfaces and curves in multi-dimensional space

1. Introduction

The problem of uniform distribution of points on parametric surfaces in multi-dimensional Euclidean space is important for various field of science such as mathematical modeling, numerical modeling, Monte Carlo techniques and many others.

In [1-3], methods for uniform distribution of points on hypersphere were presented. These methods were developed by G. Marsaglia and M. Muller. In [4], the approach for uniform distribution of points on hypersphere and also its modification for uniform distribution of points on hyperellipsoid were presented.

It's important to note that in [5] the universal algorithm for uniform distribution of points on parametric surfaces in three-dimensional spaces was already proposed and described. In this paper the method for uniform distribution of points on analytic surfaces defined by the parametric method in multi-dimensional Euclidean space is proposed. This method is generalization and development of method proposed in [5].

The proposed method can be used for uniform distribution of points on the hypersphere and hyperellipsoid (as additional one to already existing methods), and for uniform distribution of points on the other parametric surfaces in a multi-dimensional Euclidean space. Due to generality the proposed method can be also used for uniform distribution of points on curves in a multi-dimensional Euclidean space. The method also works in three-dimensional physical space which is an ordinary for the human perception, and certainly it can be applied to a uniform distribution of points on curves and surfaces in three-dimensional space for various scientific problems.

The proposed method is enough general for uniform distribution of points on various surfaces in multi-dimensional Euclidean space because it allows to distribute points uniformly on arbitrary analytic surfaces in multi-dimensional space which can be defined in the parametric form $\mathbf{r}(u_1, u_2, ..., u_m) \in R^n$, n - dimensionality of the space, m - dimensionality of the surfaces ($m < n, \quad m_{\max} = n-1$).

The proposed algorithm consists of two main parts. The first one is the finding of density function of the joint distribution of values of parameters $u_1, u_2, ..., u_m$ corresponding to uniform distribution of points on a parametric given surface. The second one is the generating of multi-dimensional random variable using generalized Neumann's method called also multi-dimensional acceptance-rejection method [4].

2. Statement of problem

Let m-dimensional surface be defined by parametric functions in n-dimensional Euclidean space as

$$\mathbf{r}(u_1, u_2, ..., u_m) = \begin{pmatrix} x_1(u_1, u_2, ..., u_m) \\ x_2(u_1, u_2, ..., u_m) \\ \cdots \\ x_n(u_1, u_2, ..., u_m) \end{pmatrix}, \tag{2.1}$$

where the parameters $u_1, u_2, ..., u_m$ are defined on the domain

$$D = \{u_1^{\min} \le u_1 \le u_1^{\max}, u_2^{\min} \le u_2 \le u_2^{\max}, ..., u_m^{\min} \le u_m \le u_m^{\max}\}. \tag{2.2}$$

It is necessary to distribute uniformly points on this surface.

3. Density function of the joint distribution of values of parameters corresponding to uniform distribution of points on a given surfaces

Let parametric surface be defined by parametric functions

$$\mathbf{r}(u_1,u_2,...,u_m)=\begin{pmatrix} x_1(u_1,u_2,...,u_m) \\ x_2(u_1,u_2,...,u_m) \\ ... \\ x_n(u_1,u_2,...,u_m) \end{pmatrix}, \tag{3.1}$$

where the parameters $u_1,u_2,...,u_m$ are defined on the domain

$$D=\{u_1^{\min}\le u_1\le u_1^{\max},u_2^{\min}\le u_2\le u_2^{\max},...,u_m^{\min}\le u_m\le u_m^{\max}\}. \tag{3.2}$$

It is necessary to find analytically a density function $f(u_1,u_2,...,u_m)$ of the joint distribution of values of parameters $u_1,u_2,...,u_m$ corresponding to uniform distribution of points on the considered surface.

In the case when points are uniformly distributed on the considered surface, according to geometrical interpretation, a probability of entering of any point C on a surface area element dA, on the one hand, can be defined as

$$P(C\subset dA)=\frac{dA}{A}, \tag{3.3}$$

where

$$dA=\sqrt{g}\,du_1du_2...du_m. \tag{3.4}$$

Here $g=\det(g_{ij})$ - determinant of the metric tensor matrix on the surface.

The matrix of the metric tensor on the surface $\mathbf{r}(u_1,u_2,...,u_m)$ has form

$$\left(g_{ij}\right)=\begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1m} \\ g_{21} & g_{22} & \cdots & g_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ g_{m1} & g_{m2} & \cdots & g_{mm} \end{pmatrix}, \tag{3.5}$$

where $i,j=1,2,...,m$, $g_{ij}=(\frac{\partial\mathbf{r}(u_1,u_2,...,u_m)}{\partial u_i},\frac{\partial\mathbf{r}(u_1,u_2,...,u_m)}{\partial u_j})$ - element of the matrix of the metric tensor on the surface, (... , ...) - denotes the operation of the scalar product of the vector functions, that is $g_{ij}=\sum_{k=1}^{n}(\frac{\partial x_k(u_1,u_2,...,u_m)}{\partial u_i}\cdot\frac{\partial x_k(u_1,u_2,...,u_m)}{\partial u_j})$. It is important to note that $g>0$.

The surface area on the domain D is equal

$$A=\iiint\limits_D...\int\sqrt{g}\,du_1du_2...du_m. \tag{3.6}$$

And consequently,

$$P(C \subset dA) = \frac{\sqrt{g}\,du_1 du_2 ... du_m}{\iiint\limits_D ... \int \sqrt{g}\,du_1 du_2 ... du_m}. \quad (3.7)$$

On the other hand, a probability of entering of any point C on a surface area element dA can be also defined as

$$P(C \subset dA) = f(u_1, u_2, ..., u_m) du_1 du_2 ... du_m, \quad (3.8)$$

where $f(u_1, u_2, ..., u_m)$ is the required density function of the joint distribution of parameters $u_1, u_2, ..., u_m$.

Taking into account expressions (3.7) and (3.8) we obtain the required density function of the joint distribution of values of the parameters $u_1, u_2, ..., u_m$ which corresponds to the uniform distribution of points on surface

$$f(u_1, u_2, ..., u_m) = \frac{\sqrt{g}}{\iiint\limits_D ... \int \sqrt{g}\,du_1 du_2 ... du_m}. \quad (3.9)$$

By generating the values of $u_1, u_2, ..., u_m$ with the help of the obtained function $f(u_1, u_2, ..., u_m)$, and then finding corresponded coordinates of points, we obtain uniform distribution of points on the considered surface.

4. Generating multi-dimensional random variables by using a known density function of the joint distribution

Various methods are used to generate values of one-dimensional random variable by using a known density function of distribution, see [4]. For example, the inverse-transform method can be applied in the cases when the probability distribution function can be obtained analytically. However, application of this method meets difficulties in the cases of multi-dimensional distributions of dependent random variables. A universal method for generating values of one-dimensional and multi-dimensional random variable is the acceptance-rejection method known also as Neumann's method [4].

Firstly, let us consider the acceptance-rejection method on the example of one-dimensional random variable values generation by using function of the joint distribution. Then we consider a generalization of this method for multi-dimensional random variable generating by using a known density function of the joint distribution.

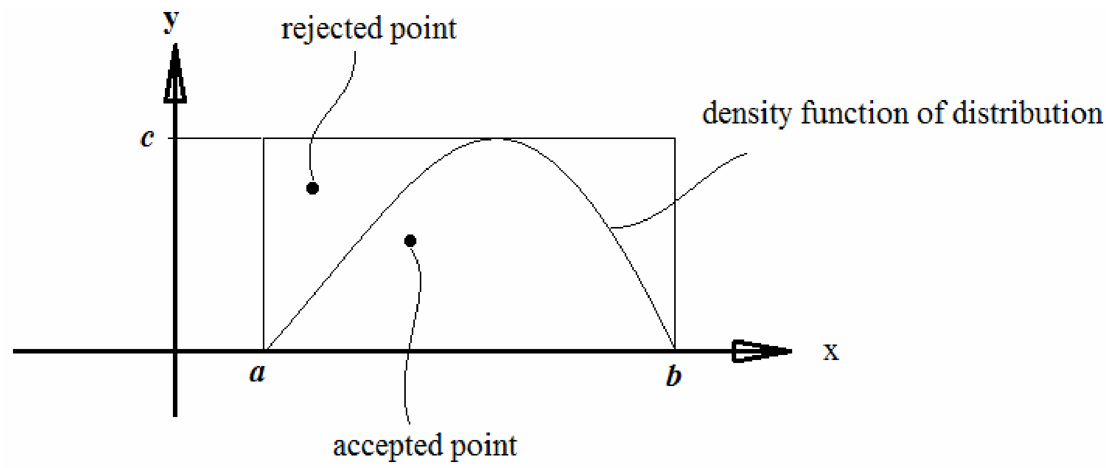

Fig. 4.1. Illustration of acceptance-rejection method for one-dimensional random variable generating with the help of the density function of distribution

In the case of generation of one-dimensional random variable values generating, the acceptance-rejection method consists in the following steps (see fig. 4.1):

1) The density function of distribution is placed in the rectangle such a way as it is shown in figure 4.1;

2) One generates random point with coordinates $x = (b - a)\text{Random} + a$, $y = c\text{Random}$, where Random is random number with uniform distribution on interval $(0,1)$;

3) The obtained point is accepted if it lies below the curve of density function of distribution. Otherwise, the point is rejected (see fig. 4.1);

4) One then repeats steps 2, 3.

In application to multi-dimensional cases the generation procedure is unchanged except one take into account changes in the dimensionality of space. For m-dimensional random variables corresponding to our case the procedure is carried out in the $(m+1)$ - dimensional space. In this case, the algorithm is implemented as follows:

1) One finds $\max\limits_D f(u_1, u_2, ..., u_m)$ - maximal value of function $f(u_1, u_2, ..., u_m)$ on the domain $D = \{u_1^{\min} \le u_1 \le u_1^{\max}, u_2^{\min} \le u_2 \le u_2^{\max}, ..., u_m^{\min} \le u_m \le u_m^{\max}\}$.

2) m random numbers $u_1^0 = (u_1^{\max} - u_1^{\min})\text{Random}+u_1^{\min}$, $u_2^0 = (u_2^{\max} - u_2^{\min})\text{Random}+u_2^{\min}$, … , $u_m^0 = (u_m^{\max} - u_m^{\min})\text{Random}+u_m^{\min}$ are generated, where Random is random number on interval $(0,1)$;

3) If $f(u_1^0, u_2^0, ..., u_m^0) > \text{Random} \cdot \max\limits_D f(u_1, u_2, ..., u_m)$, the point is accepted (here, Random is also random number on interval $(0,1)$). Otherwise, the point is rejected;

4) One repeats steps 2, 3.

It is very important to note that in this algorithm the function $f^*(u_1,u_2,...,u_m)=\sqrt{g}$ can be used instead of the function $f(u_1,u_2,...,u_m)=\dfrac{\sqrt{g}}{\iiint\limits_D ...\int \sqrt{g}\,du_1du_2...du_m}$. It is possible because

$$f(u_1^0,u_2^0,...,u_m^0) > \text{Random}\cdot\max_D f(u_1,u_2,...,u_m) \iff$$

$$\iff \left.\frac{\sqrt{g}}{\iiint\limits_D ...\int \sqrt{g}\,du_1du_2...du_m}\right/_{\substack{u_1=u_1^0,\\ u_2=u_2^0,\\ ...,\\ u_m=u_m^0}} > \text{Random}\cdot\max_D \frac{\sqrt{g}}{\iiint\limits_D ...\int \sqrt{g}\,du_1du_2...du_m} \iff$$

$$\iff \left.\sqrt{g}\right/_{\substack{u_1=u_1^0,\\ u_2=u_2^0,\\ ...,\\ u_m=u_m^0}} > \text{Random}\cdot\max_D \sqrt{g}.$$

This allows simplify the calculations.

By generating the values of $u_1,u_2,...,u_m$ with the help of density function $f(u_1,u_2,...,u_m)$ using described in this section multi-dimensional acceptance-rejection method and then finding corresponding coordinates values of coordinates of points; we obtain uniform distribution of points on the considered surface.

5. Uniform distribution of points on curves

When $m=1$ we have the curve in R^n, and the matrix of the metric tensor has form

$$\left(g_{ij}\right)=\left(g_{11}\right),$$

where

$$g_{11}=\sum_{k=1}^{n}\left(\frac{\partial x_k(u_1)}{\partial u_1}\cdot\frac{\partial x_k(u_1)}{\partial u_1}\right)=\left(\frac{\partial x_1(u_1)}{\partial u_1}\right)^2+\left(\frac{\partial x_2(u_1)}{\partial u_1}\right)^2+...+\left(\frac{\partial x_n(u_1)}{\partial u_1}\right)^2,$$

or in a more conventional form, when $u_1=t$, it can be rewritten as

$$g_{11}=\sum_{k=1}^{n}\left(\frac{\partial x_k(t)}{\partial t}\cdot\frac{\partial x_k(t)}{\partial t}\right)=\left(\frac{\partial x_1(t)}{\partial t}\right)^2+\left(\frac{\partial x_2(t)}{\partial t}\right)^2+...+\left(\frac{\partial x_n(t)}{\partial t}\right)^2.$$

Thus the density function for uniform distribution of points on curves is equal

$$f(t)=\frac{\sqrt{(\frac{\partial x_1(t)}{\partial t})^2+(\frac{\partial x_2(t)}{\partial t})^2+\ldots+(\frac{\partial x_n(t)}{\partial t})^2}}{\int\limits_{t_1}^{t_2}\sqrt{(\frac{\partial x_1(t)}{\partial t})^2+(\frac{\partial x_2(t)}{\partial t})^2+\ldots+(\frac{\partial x_n(t)}{\partial t})^2}\,dt}.$$

In three-dimensional space we respectively have

$$f(t)=\frac{\sqrt{(\frac{\partial x(t)}{\partial t})^2+(\frac{\partial y(t)}{\partial t})^2+(\frac{\partial z(t)}{\partial t})^2}}{\int\limits_{t_1}^{t_1}\sqrt{(\frac{\partial x(t)}{\partial t})^2+(\frac{\partial y(t)}{\partial t})^2+(\frac{\partial z(t)}{\partial t})^2}\,dt}.$$

In figure 5.1 uniform distribution of points on planar curve is shown. The curve is defined by parametric equations

$$x(t)=t,\ \ y(t)=-10e^{-100(t-4{,}25)^2}+\sin 6t-2\sin t+2,\ \ z(t)=0,$$

where $t\in[0,10]$.

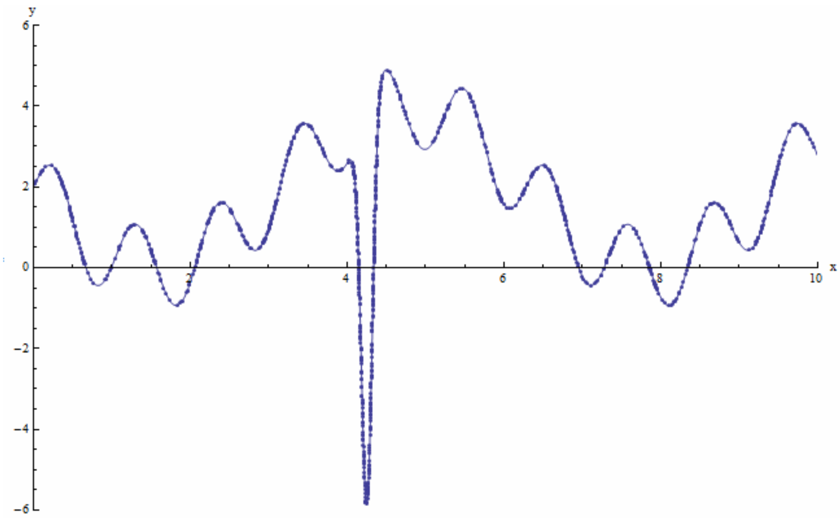

Fig. 5.1. Uniform distribution of points on planar curve

In figure 5.2 uniform distribution of points on Viviani's curve is shown. The Viviani's curve is defined by parametric equations

$$x(t)=R(1+\cos t),\ y(t)=R\sin t,\ z(t)=2R\sin\frac{t}{2},$$

where $R=1,\ t\in[0,4\pi]$.

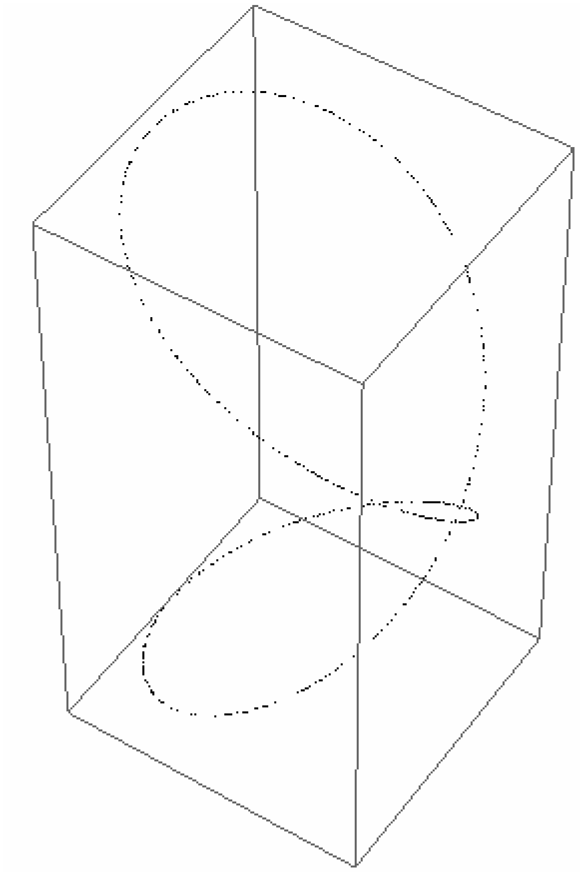

Fig. 5.2. Uniform distribution of points on Viviani's curve

6. Uniform distribution of points on surfaces in three-dimensional space

When $m = 2$ and $n = 3$ we have the surface in R^3, and the matrix of the metric tensor has form

$$\left(g_{ij}\right) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

The function $f(u_1, u_2, ..., u_m) = \dfrac{\sqrt{g}}{\iiint\limits_D ... \int \sqrt{g}\, du_1 du_2 ... du_m}$ can be written as

$$f(u_1, u_2) = \frac{\sqrt{g_{11} g_{22} - g_{12}{}^2}}{\iint\limits_D \sqrt{g_{11} g_{22} - g_{12}{}^2}\, du_1 du_2}.$$

Taking into account that $g_{11} = E$, $g_{12} = g_{21} = F$, $g_{22} = G$ are the coefficients of the first fundamental form of surface, the function (6.2) can be rewritten

$$f(u, v) = \frac{\sqrt{EG - F^2}}{\iint\limits_D \sqrt{EG - F^2}\, du dv}.$$

This function was already obtained in [5]. Thus we obtain the result of [5].

In figure 6.1 uniform distribution of points on surface of torus is shown. The surface of torus is defined by parametric equations

$$x = x(u,v) = (3+\cos u)\cos v\,,\ \ y = y(u,v) = (3+\cos u)\sin v\,,\ \ z = z(u,v) = \sin u\,,$$

where $0 \le u \le 2\pi$, $0 \le v \le 2\pi$.

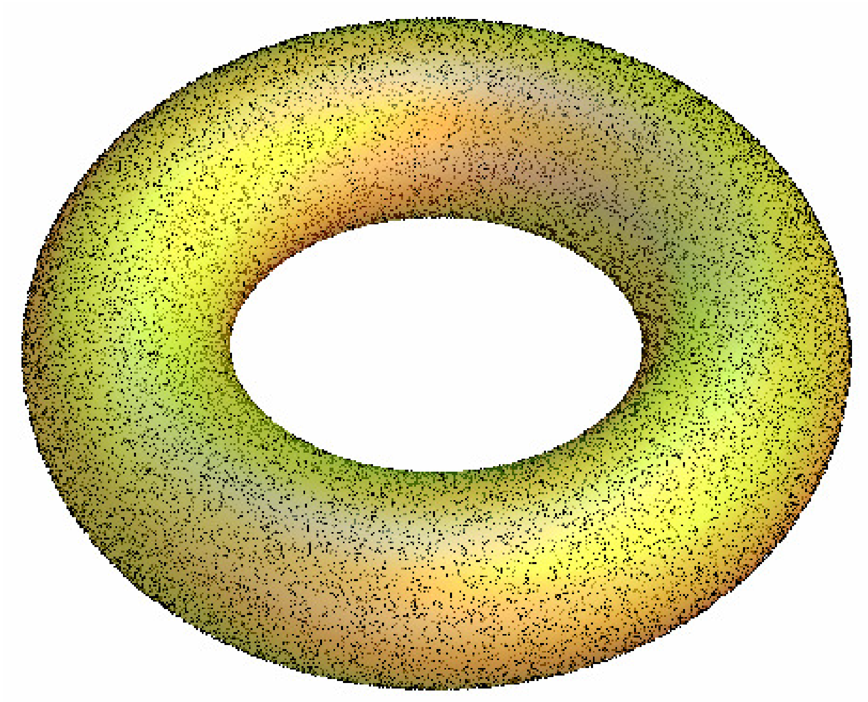

Fig. 6.1. Uniform distribution of points on surface of torus

In figure 6.2 uniform distribution of points on surface of Klein bottle is shown. The surface of Klein bottle is defined by parametric equations [6]:

$$x = x(u,v) = -\frac{2}{15}\cos u(3\cos v + 5\sin u\cos v\cos u - 30\sin u - 60\sin u\cos^6 v + 90\sin u\cos^4 v)\,,$$

$$y = y(u,v) = -\frac{1}{15}\sin u(80\cos v\cos^7 u\sin u + 48\cos v\cos^6 u - 80\cos v\cos^5 u\sin u -$$
$$-48\cos v\cos^4 u - 5\cos v\cos^3 u\sin u - 3\cos v\cos^2 u + 5\sin u\cos v\cos u + 3\cos v - 60\sin u)\,,$$

$$z = z(u,v) = \frac{2}{15}\sin v(3 + 5\sin u\cos u)\,,$$

where $-\pi/2 \le u \le \pi/2$, $0 \le v \le 2\pi$.

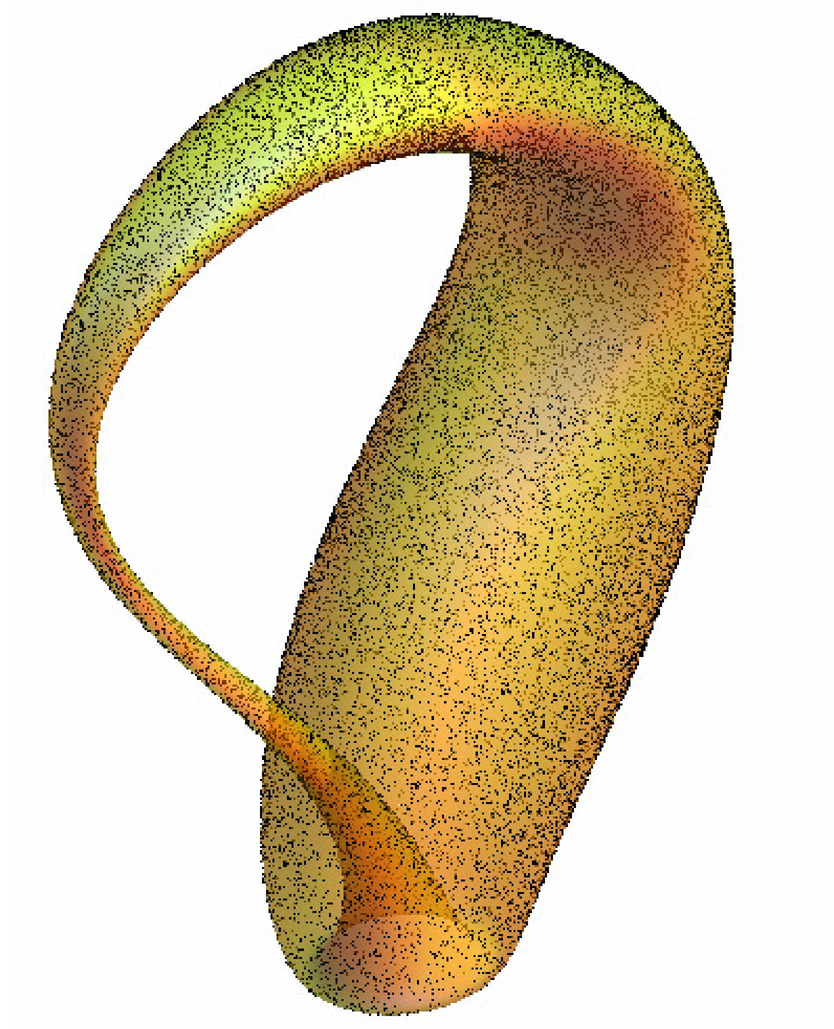

Fig. 6.2. Uniform distribution of points on surface of Klein bottle

7. Conclusions

The proposed method can be applied for uniform distributions of points on various parametric surfaces and hypersurfaces. The method of G. Marsaglia is very simple and reliable, but it can be applied only for hypersphere, while the area of possible application of the proposed method is much wider.

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